Solutions

1. No, in equivalence classes element of the equivalence class are only related to each other, i.e. elements of E_i are only related to each other.

OR

Given function $$f(x) = 2x,$$

Let for $y \in N$ (codomain), there exists $x \in N$(domain), such that

$$y = f(x) \Rightarrow y = 2x \Rightarrow x = \frac{y}{2}, \text{ may not belong to } N$$

e.g. let $y = 7$, then $x = \frac{7}{2} \notin N$. As '7' in the co-domain is not associated to any $x \in$ domain. Hence, not onto.

2. For equivalence class {1}

$(a, 1) \in R$ for $a \in A$

$\Rightarrow |a - 1|$ is a multiple of 3

$\Rightarrow a - 1 = 3\lambda \Rightarrow a = 3\lambda + 1$

$\Rightarrow$ a = 1, 4, 7, 10

$\therefore \{1\} = \{1, 4, 7, 10\}$

3. Given $f(x) = \sin x, \quad x \in [0, \pi]$

Let $x_1 = \frac{\pi}{3}$ and $x_2 = \frac{2\pi}{3}$

$$\therefore \quad f\left(\frac{\pi}{3}\right) = \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$$

$$f\left(\frac{2\pi}{3}\right) = \sin\frac{2\pi}{3} = \frac{\sqrt{3}}{2}$$

We notice $x_1 \neq x_2$ but $f(x_1) = f(x_2)$

Hence, not one-one

OR

Let A = {1, 2}

Identity relation is {(1, 1}, (2, 2)}

Reflexive relation is {(1, 1), (2, 2), (1, 2)}

We notice identity relation is a subset of reflexive relation.

Hence, identity relation is a reflexive relation but a reflexive relation may or may not be identity relation.

4. $A \cdot \text{Adj} A = |A| I$

5. Consider $$\begin{vmatrix} k & 3 \\ 4 & k \end{vmatrix} = \begin{vmatrix} 4 & -3 \\ 0 & 1 \end{vmatrix}$$

$$\Rightarrow \quad k^2 - 12 = 4 - 0 \Rightarrow k^2 = 16 \Rightarrow k = \pm 4$$

$$\Rightarrow \quad k = 4 \in N.$$

OR

Consider $$\Delta = \begin{vmatrix} a & d & g \\ b & e & h \\ c & f & i \end{vmatrix}$$

We know value of determinant is same of we take transpose

$$\therefore \quad \Delta = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = p = 20$$

6. $BA = B \Rightarrow (BA)B = BB$

$\Rightarrow \quad B(AB) = B^2$

$\Rightarrow \quad BA = B^2 \Rightarrow B = B^2$

7. Consider $\int \frac{\cos\sqrt{x}}{\sqrt{x}} dx = 2\int \cos t\, dt = 2\sin t + C = 2\sin\sqrt{x} + C$

Let $\sqrt{x} = t$

$\Rightarrow \frac{1}{2\sqrt{x}}dx = dt$

or $\frac{1}{\sqrt{x}}dx = 2dt$

OR

Consider $\int \sin 2x \sin 3x\, dx = \frac{1}{2}\int 2\sin 3x \sin 2x\, dx$

$$= \frac{1}{2}\int (\cos x - \cos 5x)\, dx$$

$$= \frac{1}{2}\left[\sin x - \frac{\sin 5x}{5}\right] + C, \text{ where } C \text{ is constant of integration.}$$

$[\because 2\sin A \sin B = \cos(A - B) - \cos(A + B)]$

8. Curve is symmetrical about the y-axis

$$\therefore \quad \text{Area} = 2\int_0^4 x\, dy = 2\int_0^4 \sqrt{y}\, dy$$

$$= \frac{4}{3}\left[y^{3/2}\right]_0^4 = \frac{4}{3} \times (8 - 0) = \frac{32}{3} \text{ sq units}$$

9. One

OR

$$\frac{dy}{dx} = e^{2x+y} = e^{2x} . e^y$$

$$\Rightarrow \quad e^{-y}dy = e^{2x}dx$$

$$\int e^{-y}\, dy = \int e^{2x}\, dx$$

$$-\frac{1}{e^y} = \frac{1}{2}e^{2x} + C$$

10. As vectors are perpendicular, then $(2\hat{i} + \lambda\hat{j} + \hat{k}).(\hat{i} - 2\hat{j} + 3\hat{k}) = 0$

$\Rightarrow \quad 2 - 2\lambda + 3 = 0 \quad \Rightarrow \quad \lambda = \frac{5}{2}$

11. Consider $|\vec{a} \times \vec{b}|^2 = (|\vec{a}||\vec{b}|\sin\theta)^2 = |\vec{a}|^2|\vec{b}|^2\sin^2\theta$

$$= |\vec{a}|^2|\vec{b}|^2(1 - \cos^2\theta) = |\vec{a}|^2|\vec{b}|^2 - |\vec{a}|^2|\vec{b}|^2\cos^2\theta$$

$$= \vec{a}^2\vec{b}^2 - (|\vec{a}||\vec{b}|\cos\theta)^2 = \vec{a}^2\vec{b}^2 - (\vec{a}.\vec{b})^2$$

$$= \begin{vmatrix} \vec{a}^2 & \vec{a}.\vec{b} \\ \vec{a}.\vec{b} & \vec{b}^2 \end{vmatrix} = \begin{vmatrix} \vec{a}.\vec{a} & \vec{a}.\vec{b} \\ \vec{b}.\vec{a} & \vec{b}.\vec{b} \end{vmatrix}$$

$[\because \vec{a}^2 = \vec{a}.\vec{a}; \vec{a}.\vec{b} = \vec{b}$

12. Position vector $= \frac{2(\vec{a} + \vec{b}) + 1(2\vec{a} - \vec{b})}{1 + 2}$

$$= \frac{4\vec{a} + \vec{b}}{3}$$

13. $$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$\frac{5}{6} = P(A) + \left[1 - \frac{1}{2}\right] - \frac{1}{3}$$

$$P(A) = \frac{5}{6} - \frac{1}{2} + \frac{1}{3} = \frac{5-3+2}{6} = \frac{4}{6} = \frac{2}{3}$$

As $$P(A) \cdot P(B) = \frac{2}{3} \cdot \frac{1}{2} = \frac{1}{3} = P(A \cap B)$$

$\therefore$ Events A and B are independent events.

14. $$P(\text{problem solved}) = 1 - P(\text{no one solves})$$

$$= 1 - \frac{1}{2} \times \frac{2}{3} = 1 - \frac{1}{3} = \frac{2}{3}$$

15. Given plane $2x - 3y + 6z = 7$

Dividing by $\sqrt{4+9+36} = 7$, we get

normal form of plane as $\frac{2}{7}x - \frac{3}{7}y + \frac{6}{7}z = 1$

$\therefore$ Direction cosines of normal to plane are $\frac{2}{7}, \frac{-3}{7}, \frac{6}{7}$.

16. Line is $\frac{4-x}{2} = \frac{y}{6} = \frac{1-z}{3}$, i.e. $\frac{x-4}{-2} = \frac{y}{6} = \frac{z-1}{-3}$. Direction ratios are –2, 6, –3 or 2, –6, 3.

17. (*i*) (*c*), Perimeter = 1000

$$\Rightarrow \quad 2y + 2x + \pi y = 1000$$

(*ii*) (*a*), Area of sports ground

$$\therefore \quad A = 2xy = 2x\left[\frac{1000 - 2x}{2+\pi}\right] \quad \text{[from (i)]}$$

$$= \frac{2}{2+\pi}(1000x - 2x^2)\text{m}^2$$

(*iii*) (*d*), at the point of maximum area

$$\frac{dA}{dx} = 0 \Rightarrow \frac{2}{2+\pi}[1000 - 4x] = 0 \Rightarrow x = 250$$

$$\Rightarrow \quad \frac{d^2A}{dx^2} < 0. \text{ for } x = 250 \text{ m}$$

(*iv*) (*a*), Total area including parking = $B(A + \text{Parking area}) = 2xy + \frac{1}{2}\pi y^2$

$$= [1000 - (2 + \pi)y]y + \frac{1}{2}\pi y^2$$

$$= 1000y - (2 + \pi)y^2 + \frac{1}{2}\pi y^2$$

$$\frac{dB}{dy} = 1000 - 2(2 + \pi)y + \pi y = 1000 - (4 + \pi)y$$

For the point of maximum area

$$\frac{dB}{dy} = 0 \Rightarrow y = \frac{1000}{4+\pi}$$

$$\frac{d^2B}{dy^2} < 0 \text{ for } y = \frac{1000}{4+\pi}$$

(*v*) (*c*), $$A_{max} = \frac{2}{2+\pi}[1000 \times 250 - 2(250)^2]$$

$$= \frac{2 \times 250}{2+\pi}(1000 - 500) = \frac{250000}{2+\pi}\text{m}^2$$

18. A : helping on regular basis

B : contributing to Prime Minister relief fund

C : helping through NGO's

E : person needs help

(*i*) (*d*), $P(C) = 1-[P(A)+P(B)] = 1-\left[\frac{50}{200}+\frac{120}{200}\right] = \frac{3}{20}$

(*ii*) (*b*), P(E/B)

(*iii*) (c), $P(E) = P(A)\cdot P(E/A) + P(B)\cdot P(E/B) + P(C)\cdot P(E/C)$

$$= \frac{5}{20}\times 0.15 + \frac{12}{20}\times 0.06 + \frac{3}{20}\times 0.10$$

$$= \frac{0.75+0.72+0.30}{20} = \frac{1.77}{20} = 0.0885$$

(*iv*) (*a*), $P(A/E) = \frac{P(A)P(E/A)}{P(E)} = \frac{0.75}{1.77} = \frac{75}{177}$

(*v*) (*a*), Number of persons $= \frac{5}{20}\times 100000$

$$= 25000$$

19.

$$\tan^{-1}(1) = \frac{\pi}{4}$$

$$\tan^{-1}(-\sqrt{3}) = -\tan^{-1}(\sqrt{3}) = -\frac{\pi}{3}$$

$$\therefore \quad \tan^{-1}1 + \tan^{-1}(-\sqrt{3}) = \frac{\pi}{4}-\frac{\pi}{3} = -\frac{\pi}{12}.$$

20.

$$2A + B + X = O \Rightarrow X = -2A - B = -2\begin{bmatrix}-1 & 2\\ 3 & 4\end{bmatrix} - \begin{bmatrix}3 & -2\\ 1 & 5\end{bmatrix}$$

$$= \begin{bmatrix}2 & -4\\ -6 & -8\end{bmatrix} + \begin{bmatrix}-3 & 2\\ -1 & -5\end{bmatrix}$$

$$= \begin{bmatrix}2-3 & -4+2\\ -6-1 & -8-5\end{bmatrix} = \begin{bmatrix}-1 & -2\\ -7 & -13\end{bmatrix}.$$

OR

$$A = \begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}; \quad kA = \begin{bmatrix}k & -k\\ -k & k\end{bmatrix}$$

$$A^2 = \begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}$$

$$= \begin{bmatrix}1+1 & -1-1\\ -1-1 & 1+1\end{bmatrix} = \begin{bmatrix}2 & -2\\ -2 & 2\end{bmatrix}$$

$$A^2 = kA \Rightarrow \begin{bmatrix}2 & -2\\ -2 & 2\end{bmatrix} = \begin{bmatrix}k & -k\\ -k & k\end{bmatrix}$$

$$\Rightarrow \quad k = 2.$$

21. The given function is

$$f(x) = \begin{cases} kx+1, & \text{if } x \le \pi \\ \cos x, & \text{if } x > \pi \end{cases}$$

The given function f is continuous at $x = \pi$.

$$\Rightarrow \quad \lim_{x \to \pi^-} f(x) = \lim_{x \to \pi^+} f(x) = f(\pi)$$

$$\Rightarrow \quad \lim_{x \to \pi^-} (kx + 1) = \lim_{x \to \pi^+} \cos x = k\pi + 1$$

$$\lim_{h \to 0} [k(\pi - h) + 1] = \lim_{h \to 0} \cos(\pi + h) = k\pi + 1$$

$$\Rightarrow \quad k\pi + 1 = \cos \pi = k\pi + 1$$

$$\Rightarrow \quad k\pi + 1 = -1 = k\pi + 1$$

$$\Rightarrow \quad k = -\frac{2}{\pi}$$

Hence, the required value of k is $\left(-\frac{2}{\pi}\right)$.

22. Consider $\qquad f(x) = \tan^{-1}(\sin x + \cos x)$

$$\Rightarrow f'(x) = \frac{1}{1 + (\sin x + \cos x)^2} \cdot (\cos x - \sin x) \qquad ...(i)$$

Sign of $f'(x)$ depends upon $(\cos x - \sin x)$

We know for $x \in \left(\frac{\pi}{4}, \frac{\pi}{2}\right)$, $\cos x < \sin x$

$\Rightarrow \cos x - \sin x < 0$

$\Rightarrow f'(x) < 0$ [from (i)]

$\therefore$ f is decreasing for $x \in \left(\frac{\pi}{4}, \frac{\pi}{2}\right)$.

23.

$$\int \frac{3ax}{b^2 + c^2x^2} dx = \frac{3a}{2c^2} \int \frac{1}{t} dt$$

Let $b^2 + c^2x^2 = t$

$\Rightarrow 2c^2x \, dx = dt$

$$= \frac{3a}{2c^2} \log|t| + K = \frac{3a}{2c^2} \log|b^2 + c^2x^2| + K$$

OR

$$f(x) = \log\left|\frac{2 - \sin x}{2 + \sin x}\right|$$

$$f(-x) = \log\left|\frac{2 - \sin(-x)}{2 + \sin(-x)}\right| = \log\left|\frac{2 + \sin x}{2 - \sin x}\right| = -\log\left|\frac{2 - \sin x}{2 + \sin x}\right| = -f(x)$$

Function is odd.

$$\therefore \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \log\left|\frac{2 - \sin x}{2 + \sin x}\right| dx = 0$$

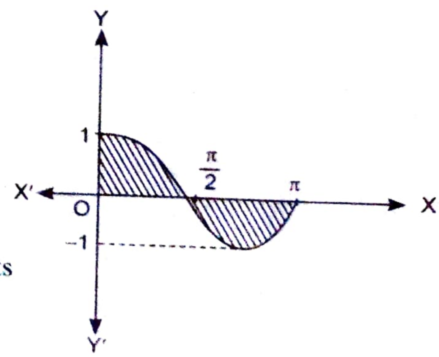
24.

$$\text{Shaded area} = \int_0^{\pi} |\cos x| \, dx$$

$$= \int_0^{\pi/2} \cos x \, dx - \int_{\pi/2}^{\pi} \cos x \, dx$$

$$= [\sin x]_0^{\pi/2} - [\sin x]_{\pi/2}^{\pi}$$

$$= \left(\sin\frac{\pi}{2} - \sin 0\right) - \left(\sin \pi - \sin\frac{\pi}{2}\right) = 2 \text{ sq units}$$



25. Consider

$$\frac{dy}{dx} = \frac{2y}{x}$$

$$\Rightarrow \quad \int \frac{dy}{y} = 2\int \frac{dx}{x}$$

$\Rightarrow \qquad \log|y| = 2\log|x| + \log C$

$= \log|x^2C|$

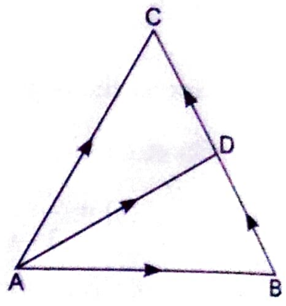
$\Rightarrow \qquad y = Cx^2 \qquad ...(i)$

Given (i) passes through (1, 1)

$\Rightarrow \qquad 1 = C\cdot(1)^2 \Rightarrow C = 1$

$\therefore$ from (i) $\qquad y = x^2$ is the required curve.

26. Given $\qquad \overrightarrow{AB} = \hat{j} + \hat{k},\ \overrightarrow{AC} = 3\hat{i} - \hat{j} + 4\hat{k}$



Using triangle law of vectors

$\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC} \Rightarrow \overrightarrow{BC} = \overrightarrow{AC} - \overrightarrow{AB}$

$\Rightarrow \overrightarrow{BC} = 3\hat{i} - \hat{j} + 4\hat{k} - \hat{j} - \hat{k} = 3\hat{i} - 2\hat{j} + 3\hat{k}$

Also $\qquad \overrightarrow{BD} = \frac{1}{2}\overrightarrow{BC}$ $\qquad$ [D is mid–point of BC]

$= \frac{3}{2}\hat{i} - \hat{j} + \frac{3}{2}\hat{k}$

In ΔABD, applying triangle law of vectors

$$\overrightarrow{AD} = \overrightarrow{AB} + \overrightarrow{BD}$$

$$= \hat{j} + \hat{k} + \frac{3}{2}\hat{i} - \hat{j} + \frac{3}{2}\hat{k} = \frac{3}{2}\hat{i} + \frac{5}{2}\hat{k}$$

Length of median $= |\overrightarrow{AD}| = \sqrt{\frac{9}{4} + \frac{25}{4}} = \frac{1}{2}\sqrt{34}$ units

27. Point through which line passes is (2, 1, – 4) and dr's: 1, – 1, – 1.

$\therefore$ Cartesian equation of line is

$$\frac{x-2}{1} = \frac{y-1}{-1} = \frac{z+4}{-1}$$

28. S : getting a total of 9 = {(3, 6), (4, 5), (5, 4), (6, 3)}

$$P(S) = \frac{4}{36} = \frac{1}{9}.\ P(\bar{S}) = \frac{8}{9}$$

A can win in 1st, 3rd, 5th, 7th, ... throws

$$P(A) = P(S) + [P(\bar{S})]^2 P(S) + [P(\bar{S})]^4 P(S) + \cdots$$

$$= \frac{1}{9} + \left(\frac{8}{9}\right)^2 \cdot \frac{1}{9} + \left(\frac{8}{9}\right)^4 \cdot \frac{1}{9} + \ldots\ldots$$

$$= \frac{\frac{1}{9}}{1 - \frac{64}{81}} = \frac{9}{17}$$

$\left[\text{sum of infinite GP } a + ar + ar^2 + \ldots = \frac{a}{1-r}\right]$

OR

As this represents a probability distribution

$\therefore \qquad \Sigma P(X) = 1 \Rightarrow 0 + 2p + 2p + 3p + p^2 + 2p^2 + 7p^2 + 2p = 1$

$\Rightarrow \qquad 10p^2 + 9p - 1 = 0 \Rightarrow 10p^2 + 10p - p - 1 = 0$

$\Rightarrow \qquad 10p(p + 1) - 1(p + 1) = 0 \Rightarrow (10p - 1)(p + 1) = 0$

$\Rightarrow \qquad 10p - 1 = 0$ or $p + 1 = 0$

$\Rightarrow \qquad p = \frac{1}{10}$ or $p = -1$ (rejected)

$\therefore \qquad p = \frac{1}{10}.$

29. A : set of all triangles and relation R is
$R = \{(T_1, T_2) \in A \times A : T_1 \sim T_2\}$.

For reflexive: For $T_1 \in A$
$(T_1, T_1) \in R \Rightarrow T_1 \sim T_1$
which is true as every triangle is similar to itself.
Hence, R is reflexive.

For symmetric: For $T_1, T_2 \in A$
$T_1 \sim T_2 \Rightarrow T_2 \sim T_1$
$\Rightarrow (T_2, T_1) \in R$ (from geometry)
Hence, R is symmetric.

For transitive: For $T_1, T_2, T_3 \in A$
Let $(T_1, T_2) \in R \Rightarrow T_1 \sim T_2$
and $(T_2, T_3) \in R \Rightarrow T_2 \sim T_3$
From geometry, we notice
$T_1 \sim T_2$ and $T_2 \sim T_3 \Rightarrow T_1 \sim T_3$
$\Rightarrow (T_1, T_3) \in R$
Hence, relation R is transitive.

As the relation R is reflexive, symmetric and transitive.
Hence, relation R is an equivalence relation.
In triangles T_1, T_2, T_3, triangles T_1 and T_3 are related as sides 3, 4, 5 and 6, 8, 10 are proportional.

30. Given $\quad y = 3at^2$ and $x = 5bt^4$

$$\frac{dy}{dt} = 6at \quad \text{and} \quad \frac{dx}{dt} = 20bt^3 \qquad ...(i)$$

$$\therefore \quad \frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = \frac{6at}{20bt^3} = \frac{3a}{10bt^2}$$

Now
$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{3a}{10bt^2}\right) = \frac{3a}{10b} \cdot \frac{d}{dx}(t^{-2}) = \frac{3a}{10b} \cdot \left(-2t^{-3} \cdot \frac{dt}{dx}\right)$$

$$= -\frac{3a}{5b} \cdot \frac{1}{t^3} \cdot \frac{1}{20bt^3} = -\frac{3a}{100b^2t^6} \qquad \text{[from } (i)]$$

$$\therefore \quad \left.\frac{d^2y}{dx^2}\right]_{t=1} = \frac{-3a}{100b^2}.$$

31. Given function $f(x) = \begin{cases} x^2 + 3x + a, & x \le 1 \\ bx + 2, & x > 1 \end{cases}$

For $x < 1$ and $x > 1$, function is a polynomial function, hence differentiable.
For 'f' to be differentiable at $x \in R$, it should be differentiable at $x = 1$.

$$\underset{x=1}{\text{LHD}} = Lf'(1) = \lim_{h \to 0} \frac{f(1-h) - f(1)}{-h} = \lim_{h \to 0} \frac{\{(1-h)^2 + 3(1-h) + a\} - \{1 + 3 + a\}}{-h}$$

$$= \lim_{h \to 0} \frac{1 + h^2 - 2h + 3 - 3h + a - 4 - a}{-h} = \lim_{h \to 0} \frac{(h^2 - 5h)}{-h} = \lim_{h \to 0} (-h + 5) = 5$$

$$\underset{x=1}{\text{RHD}} = Rf'(1) = \lim_{h \to 0} \frac{f(1+h) - f(1)}{h} = \lim_{h \to 0} \frac{\{b(1+h) + 2\} - \{1 + 3 + a\}}{h}$$

$$= \lim_{h \to 0} \frac{b + bh + 2 - 4 - a}{h} = \lim_{h \to 0} \frac{bh + b - a - 2}{h} \qquad ...(i)$$

We know if 'f' is differentiable at $x = 1$, then it is continuous at $x = 1$ also.

i.e., $\quad \underset{x=1}{\text{LHL}} = \underset{x=1}{\text{RHL}} = f(1)$

$$\lim_{x \to 1^-} (x^2 + 3x + a) = \lim_{x \to 1^+} (bx + 2) = 1 + 3 + a$$

$\Rightarrow$ $$1 + 3 + a = b + 2 = 4 + a$$

$\Rightarrow$ $$b - a - 2 = 0 \qquad \text{...}(ii)$$

Substituting in (*i*), we get

$$\underset{x=1}{\text{RHD}} = \lim_{h \to 0} \frac{bh + 0}{h} = \lim_{h \to 0} b = b \qquad \text{[from } (ii)]$$

For differentiability at $x = 1$,

$$\underset{x=1}{\text{LHD}} = \underset{x=1}{\text{RHD}}$$

$\Rightarrow$ $$5 = b$$

Substituting in (*ii*), we get

$$5 - a - 2 = 0 \Rightarrow a = 3$$

Hence, $a = 3, b = 5$ for function to be differentiable for $x \in R$.

OR

Consider, $y = (\log x)^x + x^{\log x}$

$$= e^{x \log(\log x)} + e^{\log x \cdot (\log x)} \quad [\because x^a = e^{a \log x}]$$

Now differentiating both sides, w.r.t. x, we get

$$\frac{dy}{dx} = e^{x \log(\log x)} \left\{ \log(\log x) \cdot 1 + x \cdot \frac{1}{\log x} \cdot \frac{1}{x} \right\}$$

$$+ e^{\log x \cdot \log x} \left\{ \frac{1}{x} \cdot \log x + \frac{1}{x} \cdot \log x \right\}$$

$$= (\log x)^x \left\{ \log(\log x) + \frac{1}{\log x} \right\} + x^{\log x} \cdot \left(\frac{2}{x} . \log x \right)$$

Alternatively:

Consider $y = (\log x)^x + x^{\log x}$

Let $y = u + v$

$\therefore$ $$\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} \qquad \text{...}(i)$$

Consider $u = (\log x)^x$

Taking log of both sides, we get

$$\log u = x . \log(\log x)$$

Differentiating with respect to x, we get

$$\frac{1}{u} . \frac{du}{dx} = x . \frac{1}{\log x} . \frac{1}{x} + \log(\log x) \cdot 1$$

$$\frac{du}{dx} = u \left[\frac{1}{\log x} + \log(\log x) \right]$$

$$= (\log x)^x \left[\frac{1}{\log x} + \log(\log x) \right] \qquad \text{...}(ii)$$

Consider $v = x^{\log x}$

Taking log of both sides, we get $\log v = \log x . \log x = (\log x)^2$

Differentiating with respect to x, we get $\frac{1}{v} \frac{dv}{dx} = 2(\log x) . \frac{1}{x}$

$\Rightarrow$ $$\frac{dv}{dx} = v \cdot \frac{2}{x} \log x$$

$$= x^{\log x} . \frac{2}{x} \log x \qquad \text{...}(ii)$$

Substituting from (*ii*) and (*iii*) in (*i*), we get

$$\frac{dy}{dx} = (\log x)^x \left[\frac{1}{\log x} + \log(\log x) \right] + x^{\log x} \left(\frac{2}{x} \log x \right)$$

32. Equation of the curve is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

Differentiating w.r.t. x, we get $\frac{2x}{a^2}+\frac{2y}{b^2}\frac{dy}{dx}=0 \Rightarrow \frac{y}{b^2}\frac{dy}{dx}=-\frac{x}{a^2}$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{-b^2x}{a^2y},$$

$$\left.\frac{dy}{dx}\right|_{(x_0,\,y_0)} = \frac{-b^2x_0}{a^2y_0}. \qquad \text{(slope of the tangent)}$$

Point (x_0, y_0) lies on the curve, $\frac{x_0^2}{a^2}+\frac{y_0^2}{b^2}=1$ ...(*i*)

Equation of tangent at (x_0, y_0) is $y-y_0=\frac{-b^2x_0}{a^2y_0}(x-x_0)$

$$\Rightarrow a^2yy_0-a^2y_0^2=-b^2xx_0+b^2x_0^2 \Rightarrow \frac{xx_0}{a^2}+\frac{yy_0}{b^2}=\frac{x_0^2}{a^2}+\frac{y_0^2}{b^2}$$

$$\Rightarrow \quad \frac{xx_0}{a^2}+\frac{yy_0}{b^2}=1 \qquad \text{[from (i)]}$$

33. Consider $I=\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\frac{dx}{1+\sqrt{\cot x}}$ ...(*i*)

$$I=\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\frac{dx}{1+\sqrt{\cot\left(\frac{\pi}{6}+\frac{\pi}{3}-x\right)}}$$

$$=\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\frac{dx}{1+\sqrt{\cot\left(\frac{\pi}{2}-x\right)}}$$

[using property $\int_a^b f(x)\,dx=\int_a^b f(a+b-x)\,dx$]

$$=\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\frac{dx}{1+\sqrt{\tan x}}=\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\frac{dx}{1+\frac{1}{\sqrt{\cot x}}}$$

$$=\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\frac{\sqrt{\cot x}}{\sqrt{\cot x}+1}dx \qquad \text{...(ii)}$$

Adding (*i*) and (*ii*), we get

$$2I=\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\frac{1+\sqrt{\cot x}}{1+\sqrt{\cot x}}dx=\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}1.dx$$

$$=\left[x\right]_{\frac{\pi}{6}}^{\frac{\pi}{3}}=\frac{\pi}{3}-\frac{\pi}{6}=\frac{\pi}{6}$$

$$\Rightarrow I=\frac{\pi}{12}$$

Alternatively:

$$I=\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\frac{dx}{1+\sqrt{\cot x}}dx$$

$$I=\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}dx \qquad \text{...(i)}$$

Using property $\int_a^b f(x)dx=\int_a^b f(a+b-x)dx$

$$=\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\frac{\sqrt{\sin\left(\frac{\pi}{3}+\frac{\pi}{6}-x\right)}}{\sqrt{\sin\left(\frac{\pi}{3}+\frac{\pi}{6}-x\right)}+\sqrt{\cos\left(\frac{\pi}{3}+\frac{\pi}{6}-x\right)}}dx$$

$$I=\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\frac{\sqrt{\cos x}}{\sqrt{\cos x}+\sqrt{\sin x}}dx \qquad \text{...(ii)}$$

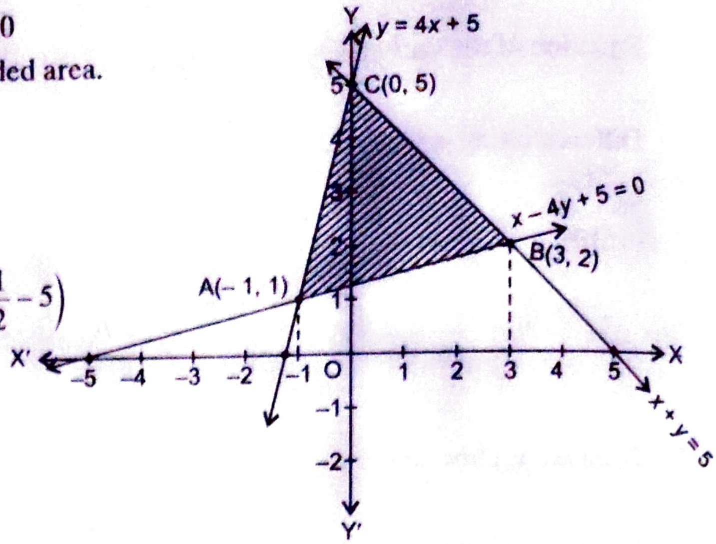
Adding (*i*) and (*ii*), we get

$$2I=\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}1\cdot dx=[x]_{\pi/6}^{\pi/3}=\frac{\pi}{3}-\frac{\pi}{6}$$

$$2I=\frac{\pi}{6}\Rightarrow I=\frac{\pi}{12}$$

34. Given lines are $y = 4x + 5$, $x + y = 5$ and $x - 4y + 5 = 0$

Plotting these on graph, we notice we have to find shaded area.



$$\text{Area} = \int_{-1}^{0}(4x+5)\,dx + \int_{0}^{3}(5-x)\,dx - \int_{-1}^{3}\frac{x+5}{4}\,dx$$

$$= (2x^2+5x)_{-1}^{0} + \left(5x - \frac{x^2}{2}\right)_0^3 - \frac{1}{4}\left(\frac{x^2}{2}+5x\right)_{-1}^{3}$$

$$= (0) - (2-5) + \left(15 - \frac{9}{2}\right) - (0) - \frac{1}{4}\left(\frac{9}{2}+15\right) + \frac{1}{4}\left(\frac{1}{2}-5\right)$$

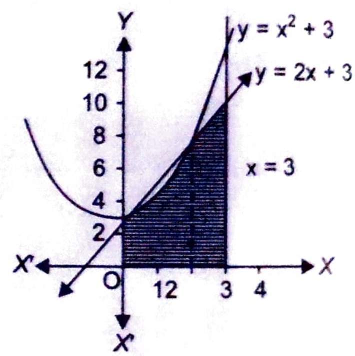
$$= 3 + \frac{21}{2} - \frac{39}{8} - \frac{9}{8} = -3 + \frac{21}{2} = \frac{15}{2} \text{ sq units}$$

OR

Region $= \{(x, y): 0 \le y \le x^2 + 3, 0 \le y \le 2x + 3, 0 \le x \le 3\}$

On plotting the inequations we have to find the area of the shaded portion.

Eliminating y from corresponding equations, we get



$$x^2 + 3 = 2x + 3$$

$$\Rightarrow \quad x = 0, 2$$

$$\therefore \quad \text{area} = \int_0^2 (x^2+3)\,dx + \int_2^3 (2x+3)\,dx.$$

$$= \left[\frac{x^3}{3} + 3x\right]_0^2 + [x^2 + 3x]_2^3$$

$$= \left(\frac{8}{3} + 6\right) - (0) + (9+9) - (4+6)$$

$$= \left(\frac{8}{3} + 6 + 18 - 10\right) \text{ sq units} = \frac{50}{3} \text{ sq units}$$

35. Consider the equation

$$y - x\frac{dy}{dx} = x + y\frac{dy}{dx}$$

$$\Rightarrow \quad y - x = (x+y)\frac{dy}{dx}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{y-x}{x+y} \qquad \text{...(}i\text{) (homogenous)}$$

Let $y = vx \Rightarrow \dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$

From (i), $\quad v + x\dfrac{dv}{dx} = \dfrac{vx - x}{x + xv} = \dfrac{v-1}{1+v}$

$$\Rightarrow \quad x\frac{dv}{dx} = \frac{v-1}{v+1} - v = \frac{v - 1 - v^2 - v}{v+1} = \frac{-(1+v^2)}{1+v}$$

$$\Rightarrow \quad \int \frac{1+v}{1+v^2}\,dv = -\int \frac{dx}{x}$$

$$\Rightarrow \quad \int \frac{1}{1+v^2}\,dv + \int \frac{v}{1+v^2}\,dv = -\int \frac{dx}{x}$$

$$\Rightarrow \quad \tan^{-1} v + \frac{1}{2}\log|1+v^2| = -\log|x| + C \qquad \left[\because \int \frac{f'(x)}{f(x)}dx = \log|f(x)| + C\right]$$

$$\Rightarrow \tan^{-1}\frac{y}{x} + \frac{1}{2}\log\left|1+\frac{y^2}{x^2}\right| = -\log|x| + C$$

$$\Rightarrow \tan^{-1}\frac{y}{x} + \frac{1}{2}\log|x^2+y^2| - \frac{1}{2}\log|x|^2 = -\log|x| + C$$

$$\Rightarrow \tan^{-1}\frac{y}{x} + \frac{1}{2}\log|x^2+y^2| = C$$ is required equation, where C is constant of integration.

36. Consider
$$A = \begin{bmatrix} 2 & 3 & 10 \\ 4 & -6 & 5 \\ 6 & 9 & -20 \end{bmatrix}$$

We have
$$A^{-1} = \frac{1}{|A|}(\text{adj } A)$$

$$|A| = \begin{vmatrix} 2 & 3 & 10 \\ 4 & -6 & 5 \\ 6 & 9 & -20 \end{vmatrix} = 2(120-45) - 3(-80-30) + 10(36+36)$$

$$= 150 + 330 + 720 = 1200 \neq 0$$

Hence, A^{-1} exists.

Matrix formed by cofactors of each element in $|A|$.

$$\begin{bmatrix} 75 & 110 & 72 \\ 150 & -100 & 0 \\ 75 & 30 & -24 \end{bmatrix}$$

$$\therefore \quad \text{Adj} A = \begin{bmatrix} 75 & 110 & 72 \\ 150 & -100 & 0 \\ 75 & 30 & -24 \end{bmatrix}' = \begin{bmatrix} 75 & 150 & 75 \\ 110 & -100 & 30 \\ 72 & 0 & -24 \end{bmatrix}$$

$$\therefore \quad A^{-1} = \frac{1}{1200}\begin{bmatrix} 75 & 150 & 75 \\ 110 & -100 & 30 \\ 72 & 0 & -24 \end{bmatrix}$$

Consider equations

$$\frac{2}{x}+\frac{3}{y}+\frac{10}{z} = 2$$

$$\frac{4}{x}-\frac{6}{y}+\frac{5}{z} = 5$$

$$\frac{6}{x}+\frac{9}{y}-\frac{20}{z} = -4$$

Corresponding matrix equation is

$$\begin{bmatrix} 2 & 3 & 10 \\ 4 & -6 & 5 \\ 6 & 9 & -20 \end{bmatrix}\begin{bmatrix} \frac{1}{x} \\ \frac{1}{y} \\ \frac{1}{z} \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ -4 \end{bmatrix}$$

$AX = B$ is matrix equation.

Its solution is $X = A^{-1}B$

Now A^{-1} is already known to us. So we can substitute and get matrix X and then x, y, z.

OR

Consider,
$$A = \begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & 2 \\ -3 & 1 & -1 \end{bmatrix} \quad ...(i)$$

We have $$A^{-1} = \frac{1}{|A|}\text{adj}A$$

$$|A| = \begin{vmatrix} 2 & 3 & 1 \\ 1 & 2 & 2 \\ -3 & 1 & -1 \end{vmatrix}$$

$$= 2(-2-2) - 3(-1+6) + 1(1+6) = -8 - 15 + 7 = -16 \neq 0$$

Hence, A^{-1} exists.

Matrix formed by cofactors of each element in $|A|$ is

$$\begin{bmatrix} -4 & -5 & 7 \\ 4 & 1 & -11 \\ 4 & -3 & 1 \end{bmatrix}$$

$$\therefore \quad \text{adj}A = \begin{bmatrix} -4 & -5 & 7 \\ 4 & 1 & -11 \\ 4 & -3 & 1 \end{bmatrix}' = \begin{bmatrix} -4 & 4 & 4 \\ -5 & 1 & -3 \\ 7 & -11 & 1 \end{bmatrix}$$

$$\therefore \quad A^{-1} = -\frac{1}{16}\begin{bmatrix} -4 & 4 & 4 \\ -5 & 1 & -3 \\ 7 & -11 & 1 \end{bmatrix} \quad ...(ii)$$

Consider equations

$$2x + y - 3z = 13$$
$$3x + 2y + z = 4$$
$$x + 2y - z = 8$$

Matrix equation is

$$\begin{bmatrix} 2 & 1 & -3 \\ 3 & 2 & 1 \\ 1 & 2 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 13 \\ 4 \\ 8 \end{bmatrix}$$

$\Rightarrow \quad A^T X = B$ [from (i)]

$\Rightarrow \quad X = (A^T)^{-1}B$ is its solution

$\Rightarrow \quad X = (A^{-1})^T B$ ..(iii)

Now we have A^{-1} [from (i)] and use $(A^T)^{-1} = (A^{-1})^T$ i.e. we take transform of A^{-1} obtained and substitute in (iii) to get X and then x, y, z.

37. If lines $\vec{r} = (2\hat{i} + \hat{j} - 3\hat{k}) + \lambda(\hat{i} + 2\hat{j} + 5\hat{k})$ and $\vec{r} = (3\hat{i} + 3\hat{j} + 2\hat{k}) + \mu(3\hat{i} - 2\hat{j} + 5\hat{k})$ lie in plane, then normal to plane is

$$\vec{n} = (\hat{i} + 2\hat{j} + 5\hat{k}) \times (3\hat{i} - 2\hat{j} + 5\hat{k})$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & 5 \\ 3 & -2 & 5 \end{vmatrix} = 20\hat{i} + 10\hat{j} - 8\hat{k}$$

$\therefore$ Plane is $[\vec{r} - (2\hat{i} + \hat{j} - 3\hat{k})] \cdot (20\hat{i} + 10\hat{j} - 8\hat{k}) = 0$

$\Rightarrow \vec{r} \cdot (20\hat{i} + 10\hat{j} - 8\hat{k}) - (40 + 10 + 24) = 0$

$\Rightarrow \vec{r} \cdot (20\hat{i} + 10\hat{j} - 8\hat{k}) - 74 = 0 \Rightarrow \vec{r} \cdot (10\hat{i} + 5\hat{j} - 4\hat{k}) - 37 = 0$ is required equation.

Equation in Cartesian form is $10x + 5y - 4z - 37 = 0$

OR

General equation of the plane passing through the point (3, 2, 0) is

$$a(x-3) + b(y-2) + c(z-0) = 0 \quad ...(i)$$

Plane (i), contains the line $\dfrac{x-3}{1} = \dfrac{y-6}{5}; z = 2$

i.e. $\qquad \dfrac{x-3}{1} = \dfrac{y-6}{5} = \dfrac{z-2}{0}$

$\therefore \quad a(3-3) + b(6-2) + c(2-0) = 0$ $\qquad$ [$\because$ points (3, 6, 2) on line, lies on plane (*i*)]

$\Rightarrow \quad 0a + 4b + 2c = 0$

$\Rightarrow \quad 0a + 2b + c = 0$ $\qquad$ ...(*ii*)

and $\quad a\cdot 1 + b\cdot 5 + c\cdot 0 = 0$ $\qquad$ ...(*iii*)

Eliminating a, b, c from (*i*), (*ii*), (*iii*), we get equation of plane as

$$\begin{vmatrix} x-3 & y-2 & z \\ 0 & 2 & 1 \\ 1 & 5 & 0 \end{vmatrix} = 0$$

$\Rightarrow \quad (x-3)(-5) - (y-2)(-1) + z(-2) = 0$

$\Rightarrow \quad 5x - 15 - y + 2 + 2z = 0$

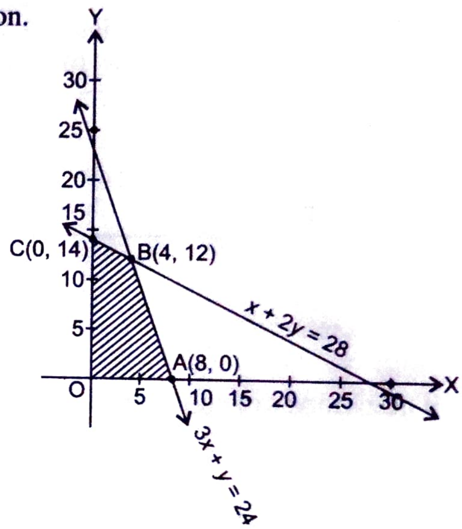
$\Rightarrow \quad 5x - y + 2z - 13 = 0$ is equation of plane.

38. Plotting the inequations on graph, we notice shaded area is feasible solution.

Possible points for maximum Z are $A(8, 0)$, $B(4,12)$, $C(0, 14)$.

Points	$Z = 20x + 40y$	Values	
$A(8, 0)$	160 + 0	160	
$B(4, 12)$	80 + 480	560	← Maximum
$C(0, 14)$	0 + 560	560	← Maximum

Z is maximum for $B(4, 12)$ or $C(0, 14)$, i.e. $x = 4, y = 12$



OR

(*i*) BC passes through (40, 0) and (0, 20)

$\therefore$ Equation is $\dfrac{x}{40} + \dfrac{y}{20} = 1 \Rightarrow x + 2y = 40$

As region contains origin (0, 0)

$\therefore \qquad x + 2y \le 40$

Also for AB inequation is

$$2x + y \le 50$$

Also $x \ge 0, y \ge 0$.

Therefore, constraints are

$$x \ge 0, \quad y \ge 0$$
$$x + 2y \le 40$$
$$2x + y \le 50$$

(*ii*) Coordinates of A, B, C are $A(25, 0)$, $B(20, 10)$, $C(0, 20)$

$$Z_A = 25 + 0 = 25$$
$$Z_B = 20 + 10 = 30 \leftarrow \text{Maximum}$$
$$Z_C = 0 + 20 = 20$$

$\therefore$ Maximum Z is at $B(20, 10)$ i.e. $x = 20, y = 10$